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The Predictive Power of the French Market Volatility Index: A Multi Horizons Study ⋆

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Abstract. The main purpose of this paper is to examine empirically the time series properties of the French Market Volatility Index (VX1). We also examine the VX1's ability to forecast future realized market volatility and finds a strong relationship. More importantly, we show how the index can be used to generate volatility forecasts over different horizons and that these forecasts are reasonably accurate predictors of future realized volatility.

Key words: market volatility index, maximum likelihood estimation, stochastic volatility models.

JEL classification codes: G14, C53, C13.

Following the CBOE (Chicago Board of Exchange) Market Volatility Index (VIX), the MONEP (Marché des Options Négociables de Paris) created, on the 8th October 1997, two volatility indexes (VX1 and VX6), based on implied volatilities of around at-the-money CAC40 Index option (PX1). VX1 is an average of four CAC 40 call option implied volatilities. While Flemming, Ostdiek and Whaley (1995) have studied the predictive power of the CBOE Market Volatility Index (VIX), the main purpose of this paper is to examine empirically the time series properties of the French Market Volatility Index.

On the basis of the evidence reported in the study of Fleming, Ostdiek and Whaley (1995), market volatility indexes which are an average of index option implied volatilities indeed appear to be useful proxies for expected market volatility. This interpretation of the volatility index may appear inconsistent with the option valuation model which determines the market volatility index. For example, in a stochastic volatility model *à la* Hull and White (1987), Feinstein (1992) demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of the option. In this paper, we suggest, in a Hull–White setting, a new methodology based on Feinstein's definition of theoretical implied volatilities in order to estimate the unobservable volatility process parameters. We apply the maximum likelihood estimation procedure on a VX1 time series

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to recover Hull and White's (1987) implicit instantaneous volatility.[1] We then compare this methodology with that of Heynen, Kemna and Vorst (1994), who used Black–Scholes implicit volatilities as proxies and with Renault and Touzi's (1996) statistical iterative procedure of filtering (of the latent volatility process) and estimation (of its parameters). In that case, with implied volatility used in the same spirit as yield to maturity on the bond market, we apply the direct maximum likelihood statistical inference as Pearson and Sun (1994) and Duan (1994) have done in the case of interest rates. Many stochastic volatility models have been proposed in the literature. In this article we assume that the variance follows a "square-root" diffusion process. This mean-reverting feature is attractive for several reasons. First, Day and Lewis (1993) empirically show that volatility shocks are persistent and mean-reverting. Second, the relationship between the spot volatility and the long-run volatility can be examined directly. Finally, this process is analytically tractable. For the price of a discount bond, Cox, Ingersoll and Ross (1985) implicitly solve for the moment-generating function of the average of this process in the derivation of their formula. Ball and Roma (1994) use this result to derive a simple closed-form expression for the expected value of average future volatility. We then show how the index can be used to generate volatility forecasts over different horizons and that these forecasts are reasonably accurate predictors of future realized volatility.

The rest of the paper is organized as follows: Section 1 explores the informational content of the French Market Volatility Index (VX1). Section 2 is devoted to the implicit maximum likelihood estimation of an Hull and White's (1987) stochastic volatility model from VX1 in order to generate volatility forecasts over different horizons.

[1] Practical use of stochastic volatility models requires a preliminary estimation of the parameters of the unobservable latent volatility process. Following Ghysels, Harvey and Renault (1996), two kinds of studies have covered this issue. First, numerous authors suggested to make inference from the observed asset price through an approximation of the structural stochastic volatility model due to an Euler or ARCH type discretization (Nelson (1990)). These models have been estimated in a variety of ways including simple Method of Moment (MM) by Taylor (1994), Generalized Methods of Moments (GMM) by Anderson and Sørensen (1994]), various Simulated Method of Moment procedures (SMM) by Duffie and Singleton (1994); Quasi Maximum Likelihood Estimation (QMLE) by Harvey, Ruiz and Shephard (1994), Simulated Maximum Likelihood Estimation (SMLE) by Danielson (1994); Indirect Inference by Gouriéroux, Montfort and Renault (1993), Moment Matching Approach by Gallant and Tauchen (1996); Bayesian Markov Chain Monte-Carlo Analysis (MCMC) by Jacquier, Polson and Rossi (1994). Apart from MM, GMM and QML these approaches are computationally intensive. Second, some recent works suggest to use option implied volatilities to get approximate data about the unobserved volatility, the observed asset being considered as exogenous.

1. VX1 as a One-Month Forecast of Stock Market Volatility

1.1. STATISTICAL PROPERTIES OF VX1

The method used by the MONEP[2] to compute the VX1 and the VX6 indexes is based on observing a quasi-line ar relationship between the premium and the volatility of the series around the at-the-money point, i.e., the most liquid series. The method used by the MONEP includes five stages. Let S_t be the price of the CAC 40 at date t and n be the number of days used in the calculation ($n = 31$ for the short-term index, VX1, and $n = 185$ for the long-term index, VX6). The aim of the calculations is to establish, at t, the implied volatility of a "virtual" at-the-money contract (i.e., the strike price is equal to the index S_t) with a constant time to maturity of n days. Since strick prices are set at the standard 25-point intervals, options are almost never at the money. Consequently, linear interpolation is used to estimate the data. The first stage consists in identifying the two nearest expiry months, being one on each side of the calculation period, n. Let τ_1 and τ_2 be the residual time to maturity (in days) corresponding to these two expiries. The next stage consists in enclosing the last price of the CAC 40 index by two strike prices, which are written K_a and K_b. Based on these two expiries τ_1 and τ_2, and the two strike prices K_a and K_b, the following four options series: (K_a, τ_1), (K_b, τ_1), (K_a, τ_2) and (K_b, τ_2) are obtained. Stage three consists in computing the value of two synthetic options with a residual life t and strike prices K_a and K_b: $C^*(K_a, n)$ and $C^*(K_b, n)$. By interpolating these synthetic values the MONEP then calculates the final value $C^{**}(S_t, n)$. The final calculation gives the price of an at-the-money option with a maturity n. It is used to obtain implied volatility. The volatility index is simply the implied volatility of the synthetic value C^{**}. To solved for implied volatility the MONEP suggests the use of the bionomial model adjusted for the daily dividends for each option contract with n periods to the expiration date from time t. Since the implied volatilities of the PX1 option series used in computing VX1 are stated in calender days (rather than in trading days), the return variance over a weekend should be three times greater than it is over any other pair of trading days. However, on empirical evidence, weekend volatility is approximately the same as the volatility during trading days. For this reason, each VX1 day is adjusted to a trading day basis by multiplying the ratio of the square root of the number of calender days, 31, to the square root of the number of trading days, 22.

Historical data is available from the MONEP WEB-site on VX1 since the beginning of 1994 through April 1998. Figure 1 plots the daily closing Volatility Index levels versus the CAC 40 Index levels. Over the sample period studied, the Market Volatility did not drift in one direction or another. Moreover, during this period spikes in the MONEP Market Volatility Index, VX1, are usually accompanied by large movements, up or down, in the stock Index level. The December 1997 Asian crisis is accompanied by more than a 50% level of Market Volatility Index.

[2] A complete and simple description is available from the MONEP WEB-site: http://issy.integra.fr:8080/monep/top_frames.htx?d=http://wwwmonep.fr/monep1_navig_111.htm.

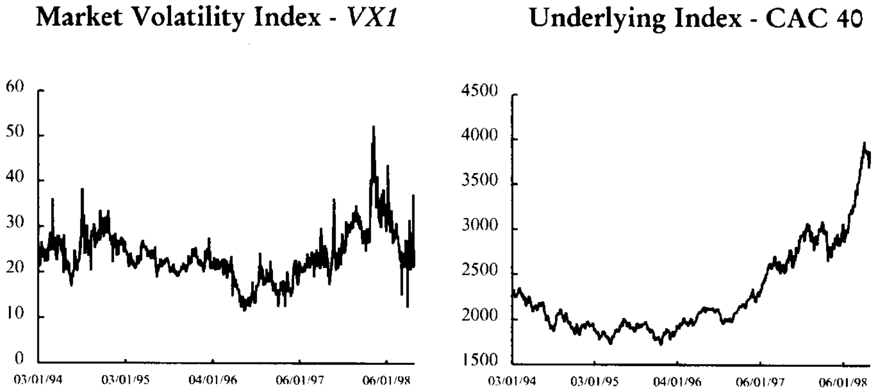

Figure 1. The daily closing volatility index and CAC 40 index.

Following Fleming, Ostdiek and Whaley (1995) the empirical analysis of the Volatility Index in this paper is done on volatility changes. Table I summarizes the properties of daily VX1 changes. The mean volatility change over the entire sample period is 0.00175. The standard deviation of the volatility changes seems high, i.e., 2.039. Table I also provides the auto-correlation structure of the Volatility Index, based on volatility changes from one through three lags. The first and second order coefficients, respectively −0.270 and −0.112 reveal a significant negative auto-correlation. This degree of correlation is similar to the auto-correlation reported by Harvey and Whaley (1991) for individual S&P100 options. They reported for daily volatility changes implied by the nearby at-the-money call (respectively put) an auto-correlation structure of −0.33 and −0.13 (respectively −0.33 and −0.09). However, this degree of correlation is much higher than the auto-correlation reported by Fleming, Ostdiek and Whaley (1995) for CBOE Market Volatility Index changes, −0.073 and −0.104. This higher auto-correlation for VX1 relative to VIX may be attributable to the difference in how the indexes are constructed. *Indeed, VX1 is a weighted average of the volatilities implied by only call PX1 option prices, whereas VIX is a weighted average of four call and four put option prices*. In fact, a call (put) implied volatility computed from the reported index level during a rising market can be upward or downward biased. Since the upward (downward) bias of the call implied volatility is approximately equal to the downward (upward) bias of the put implied volatility, the effect of infrequent trading of index stocks on the level of VIX is mitigated. Consequently, VIX Index construction reduces the oriented overreaction. Most notably the auto-correlation for one through three lags is negligible for daily CAC 40 Index returns.

1.2. INFORMATIONAL CONTENT OF VX1

Early studies of the information content in option prices focused on volatility. While Jorion (1995) and Fleming, Ostdiek and Whaley (1995) found that im-

Table I. Statistical properties of daily closing MONEP market volatility index level changes and CAC 40 index returns

Series	Statistics	Results
Volatility index changes	Mean	0.0017476
	Standard deviation	2.0391567
	Autocorrelation (ρ = 1)	−0.27018145*
	Autocorrelation (ρ = 2)	−0.1123194*
	Autocorrelation (ρ = 3)	−0.02991091
CAC 40 index returns	Mean	0.00048822
	Standard deviation	0.01127158
	Autocorrelation (ρ = 1)	−0.00362296
	Autocorrelation (ρ = 2)	0.02580488
	Autocorrelation (ρ = 3)	−0.03153352

*Identifies correlation significant at the 5% level where the standard error is calculated as $T^{-0.5}$.

plied volatilities contained substantial information for future volatility, Canina and Figlewski (1993), however, reported that implied volatilites had little predictive power for future volatility and therefore they were significantly biased forecasts. Nevertheless, this last study differs from the others in the way implied volatility is applied. Following Jorion (1995) and Fleming, Ostdiek and Whaley (1995), let $\sigma_t(\tau)$ be the realized volatility over the remaining contract life, measured from day t to day $t + \tau$ ($\tau = 1$ month), defined as:

$$\sigma_t^2(\tau) = \frac{1}{2}\left(\frac{\Gamma\left(\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau}{2}\right)}\right)^2 \sum_{i=1}^{\tau}(R_{t+i} - \bar{R})^2,$$

where R_t is the return at t and $\bar{R}$ is the sample mean of R_t. The predictive power of a volatility forecast can be estimated by regressing the realized volatility on forecast volatility:

$$\sigma_t(\tau) = a + b\hat{\sigma}_t + \epsilon_{t+1}, \tag{1}$$

where $\hat{\sigma}_t$ is the volatility forecast measured on day t, taken as the implied volatility, $\mathrm{VX1}_t$. In this case, we would expect the intercept to be zero and the slope coefficient to be unity. Canina and Figlewski (1993) suggest that an AR(1) specification using the historical rate dominates the OEX implied volatility as a forecast of realized volatility. The historical rate is defined as $\sigma_{t-\tau}(\tau)$. As pointed out by Jorion (1995), with horizons of up to one month, however, using daily data might introduce overlaps in the error terms, which causes a downward bias in the usual Ordinary Least Squares (OLS) standard errors. While White's (1980) covariance matrix presumes that the residuals to the estimated equation are serially uncorrelated, Newey and West (1987) have proposed an alternative that gives consistent estimates of the covariance matrix in the presence of both heteroscedasticity and autocorrelation. A related issue is the information content of the daily implied volatility for volatility over the next day, which can be tested using the following regression (see Jorion 1995):

$$\sqrt{R_{t+1}^2} = \alpha + \beta\hat{\sigma}_t + \epsilon_{t+1}. \tag{2}$$

Since the forecast horizon does not match with the horizon of the realized returns we only require the slope coefficient to be positive and not necessarily unity.

We first tested the information content of VX1 for next day volatility with regression Equation (2). Results are presented in Table II. The table shows that daily VX1 contains a substantial amount of information for volatility over the next day. Both slope and intercept are significant as the t-statistics, not reported in the table, are 4.55 and 2.34 respectively. Although forecasting using historical volatility also provides significant results, not only does the VX1 approximation appear less biased than historical volatility, but also its explanatory power, in terms of adjusted R^2, is higher. Finally the last line gives the implied volatility against the historical one. The low result for "historical" slope is consistent with the results of Jorion (1995) and it indicates in the same way that combining both forecasts in the same regression we can find that the historical coefficient drops to -0.001 and becomes insignificant. The predictive power hypothesis has been tested with the results presented in Table II. It shows that VX1 contains a substantial amount of information for future volatility. The slope coefficient is significantly higher than zero. These results are in sharp contrast with those of Canina and Figlewski (1993) who reported a slope coefficient of implied volatility of 0.229. The relative information content on future volatility from historical ones shows that the coefficient falls to 0.157 and thus becomes insignificant.

2. Market Volatility Index and Stochastic Volatility Models: A Multi Horizons Forecast of Stock Market Volatility

2.1. PRELIMINARY ESTIMATION OF VX1 DYNAMICS

The data generating process used is defined on a probability space (Ω, F, P) of the underlying asset price process S that is described by:

$$\frac{\mathrm{d}S}{S} = \mu(t, S, \sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W_1(t),$$

$$\mathrm{d}\sigma^2 = \kappa(\vartheta - \sigma^2)\mathrm{d}t + \gamma\sigma\,\mathrm{d}W_2(t),$$

where $W = (W_1, W_2)$ is a standard bi-dimensional Brownian motion. We denote by r the instantaneous interest rate supposed to be constant, so that the price of a zero

Table II. Tests of predictive power of VX1

Intercept	Slopes on Implied	Slopes on Historical	R^2
Information content regressions ($\sqrt{R^2_{t+1}} = \alpha + \beta\hat{\sigma}_t + \epsilon_{t+1}$)			
0.002840*	0.566668*		0.035984
(0.001213)	(0.124430)		
0.005028*		0.340435*	0.018957
(0.000831)		(0.079598)	
0.002843*	0.567985*	−0.001466	0.035089
(0.001194)	(0.168666)	(0.101194)	
Predictability regressions ($\sigma_t(\tau) = \alpha + \beta\hat{\sigma}_t + \epsilon_{t+1}$, $\tau = 1$ month)			
0.004815*	0.598552*		0.227294
(0.000874)	(0.087815)		
0.006331*		0.432168*	0.176987
(0.000710)		(0.062468)	
0.004569*	0.457014*	0.157469	0.237447
(0.000898)	(0.136825)	(0.095458)	

*Significantly different from zero at the 5 percent level.

coupon bond maturing at time T is given by $e^{-r(T-t)}$. Let C be the price process of a European call option on the underlying asset S with strike price K and maturity T. We introduce the variable $x = \ln(S/Ke^{-r(T-t)})$, and then the call option is said to be in-the-money if $x > 0$, out-of-the-money if $x < 0$, at-the-money forward if $x = 0$ and at-the-money if $x = r(T-t)$.

Following Hull and White (1987) we impose the assumption of nonsystematic volatility risk and the risk neutral data generating bivariate process is then given by:

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_1(t),$$

$$\mathrm{d}\sigma^2 = \kappa(\vartheta - \sigma^2)\,\mathrm{d}t + \gamma\sigma\,\mathrm{d}\tilde{W}_2(t),$$

where $\tilde{W} = (\tilde{W}^1, \tilde{W}^2)$ is a standard bi-dimensional Brownian motion under the risk neutral probability with $\tilde{W}_2 = W_2$.

The Hull–White formula is given by:

$$C(S, \sigma^2) = E\left[C^{\mathcal{BS}}\left(S, \frac{1}{T-t}\int_t^T \sigma_u^2\,\mathrm{d}u\right)\right] = \int C^{\mathcal{BS}}(S, u/T-t) f(u)\,\mathrm{d}u,$$

where f is the density of the cumulated variance. In this article we have assumed that the variance follows a “square-root” diffusion process which is analytically tractable. Cox, Ingersoll and Ross (1985) implicitly solve for the moment-generating function of the average of this process in the derivation of their formula for the price of a discount bond. Ball and Roma (1994) show that when there is no correlation between innovations in security price and volatility, the characteristic function of the average variance of the price process plays a pivotal role. In fact they note that this function can be used in two ways: first, to obtain the joint terminal density of the average variance and the future security price; second to obtain moments of the average variance. The first use allows for option pricing through the Fourier inversion method (see Stein and Stein (1991)) and the second use permits power series expansion methods (see Hull and White (1987)).

Along the line of Heynen, Kemna and Vorst (1994), this approach is only concerned with a certain volatility whose maturity is chosen sufficiently small enough to be considered as a proxy of the instantaneous volatility. It is then sufficient to identify both dynamics. Consider that we model the instantaneous variance as a "square-root" process:

$$\mathrm{d}\sigma^2 = \kappa(\vartheta - \sigma^2)\,\mathrm{d}t + \gamma\sigma\,\mathrm{d}\tilde{W}_2(t),$$

with $\Theta = (\kappa, \vartheta, \gamma)$. This diffusion can then be estimated by the distribution free estimation method called, by Hansen (1982), the "Generalized Method of Moments" (GMM). However as pointed out by Honoré (1997) "*The disadvantage of the GMM restrictions is that the transition density is ignored*". And indeed a maximum likelihood based procedure if possible, is preferable. A heuristic argument may be that a density function is an infinite sum of moments. If the conditional density $f(\sigma_i^2|\sigma_{i-1}^2)$ is known, then it is possible to maximize the likelihood function $l(\sigma_0^2, \ldots, \sigma_T^2; \Theta)$ in order to estimate: $\Theta = (\kappa, \vartheta, \gamma)$. The log-likelihood is then given by:

$$l(\sigma_0^2, \ldots, \sigma_T^2; \Theta) = \ln \Pi_{i=1}^{T} f(\sigma_i^2|\sigma_{i-1}^2) = \sum_{i=1}^{T} \ln f(\sigma_i^2|\sigma_{i-1}^2).$$

It's worthwhile to note that where no analytical expression of the transition density exists, the quasi-maximum likelihood estimation is useful. That is to say we apply the maximum likelihood estimation as if the process is normal. A comparison between these two estimation alternatives (GMM and QML) accomplished by Honoré (1997) on a "square-root" type diffusion concludes that: "*GMM gives more biased estimates than QML in most situations*". Jacquier, Polson and Rossi (1994) argue that "*although feasible, GMM methods are rarely used to estimate ARCH models because the likelihood function is simple to evaluate*". However, even if the QML method gives "better" results than GMM does, it possesses a discretization bias that has to be corrected. Indirect inference (see Gouriéroux, Monfort and Renault (1993)) is then recommended if the time interval between two observations is equal to one month or one week. This error is however very small if

Table III. Estimation of the VX1 dynamic

$d\sigma^2 = \kappa(\vartheta - \sigma^2)\,dt + \gamma\sigma\,dW_2(t)$		
	Value	t-stat
κ	21.5234	8.90338
ϑ	0.059076	15.5108
γ	0.676479	112.510
$\mathcal{L}$	3529.81	

the time length between two observations is short, for example a day. Nevertheless, in our case, f is a non-central χ^2 distribution:

$$\log f(\sigma_i^2|\sigma_{i-1}^2, \Delta) = \log c - c(\sigma_i^2 + e^{-k\Delta}\sigma_{i-1}^2) + \frac{1}{2}q \log\left(\frac{\sigma_i^2}{\sigma_{i-1}^2 e^{-k\Delta}}\right) + \log I_q(2c\sqrt{\sigma_i^2\sigma_{i-1}^2 e^{-k\Delta}}),$$

where $c = 2\kappa/\gamma^2(1 - e^{-\kappa\,\Delta t})$, $q = 2\kappa\vartheta/\gamma^2 - 1$, $I_q(\cdot)$ is a modified Bessel function of the first kind of order q:

$$I_q(z) = \left(\frac{z}{2}\right)^q \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n}}{n! \cdot \Gamma(q + n + 1)}.$$

Table III reports the maximum likelihood parameter estimates of the previous volatility process and the corresponding asymptotics standard errors. The period sampled is from January 1994 through April 1998. This procedure can be compared to the method used by Heynen, Kemna and Vorst (1994) who considered near-the-money, short maturity Black–Scholes implicit volatilities as proxies.

The estimate for κ, the adjustment speed for σ^2, is 21.52, which implies a very fast mean-reversion. To get a feeling of the adjustment speed we can use the following conditional expectation:

$$E[\sigma_s^2|\sigma_t^2] = \sigma_t^2 e^{-\kappa(s-t)} + \vartheta(1 - e^{-\kappa(s-t)}).$$

For example, the half-life of the process (the time when the variance is expected to have a value halfway between the current level and the long-run mean) is $\ln(2)/\kappa = 0.0322$ or about one week. It is worthwhile to note that if we had used Heynen–Kemna–Vorst's methodology, this result for adjustment speed would have been too quick relative to previous studies (see, for example, Bates (1996)). Finally it should be noted that all parameter estimates are significant. Figure 2 reports the implicit

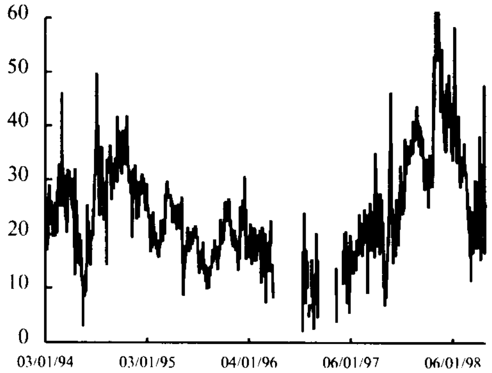

Figure 2. Implicit instantaneous volatility from market volatility index VX1.

instantaneous volatility obtained by solving an Hull and White's implicit volatility with the previous parameters.

From Figure 2 you can see that we cannot obtain all implicit instantaneous volatilities based on the annualized French Market Volatility Index and the previous estimates of the parameters. Consequently, a separate study of the dynamics in the sense that we have not taken into account the maturity effect could be a criticism. First, by considering two different proxies it implies a different result on estimation. Second implied volatility is just an approximation of the instantaneous volatility. We naturally observe an estimation bias since spot instantaneous volatility is known to be an instantaneous implied volatility.

2.2. NEW SIMPLE PROCEDURE TO CORRECT MATURITY INDUCED BIAS

Suppose that the observed prices are given by the previous formula, Renault and Touzi (1996) pointed out that, in this context, due to the increasing feature of the Black–Scholes formula with respect to the volatility parameter, a precise definition of the Black–Scholes' implied volatility can be given as the unique solution of:

$$\sigma_I^2(x, \sigma^2; \Theta) = h(x, \sigma^2; \Theta),$$

where $h = (C^{\mathcal{BS}})^{-1} \circ C$ and $\Theta = (\kappa, \vartheta, \gamma)$. In the special case of an at-the-money implied volatility, $x_t = r(T - t)$, this equation reduces to:

$$\sigma_I^2(\sigma^2; \Theta) = h(\sigma^2; \Theta).$$

The vector of parameters Θ finally need to be estimated.

Since the derivatives with respect to the volatility is positive Black–Scholes's implied volatility is a one to one function of the unobservable volatility. Thus, the observation of a Black–Scholes implied volatility is equivalent to the observation of a realization of the volatility process. In this case, since implied volatility is used in the same spirit as yield to maturity on the bond market, direct maximum likelihood statistical inference as Pearson and Sun (1994) and Duan (1994) did in the case of interest rate, may be applied. More precisely, if we denote for $i = 0, 1, \dots, n$, $\sigma^2_{I,i}$, the ith discrete and sample observation in an available time series of Market Volatility Index i.e., at-the-money implied volatility and if the conditional density for $\sigma^2_{I,i}$ is known conditionally on previous instant and noted $f(\sigma^2_{I,i}|F_{i-1}; \Theta) = f(\sigma^2_{I,i}|\sigma^2_{I,i-1}; \Theta)$ then standard maximum likelihood estimate of Θ can be obtained by using the following direct log-likelihood function:

$$\mathcal{L}(\sigma^2_{I,n}, \dots, \sigma^2_{I,0}; \Theta) = \sum_{i=1}^{n} \ln f(\sigma^2_{I,i}|\sigma^2_{I,i-1}; \Theta).$$

Since $\sigma^2_I = h(\sigma^2; \Theta)$ then the log-likelihood can be expressed as (Pearsun and Sun (1994) and Duan (1994)):

$$\mathcal{L}(\sigma^2_{I,n}, \dots, \sigma^2_{I,0}; \Theta) = \sum_{i=1}^{n} \{-\ln|J_i| + \ln f(\hat{\sigma}_i^2|\hat{\sigma}^2_{i-1}; \Theta)\}.$$

where $J_i = \partial\sigma^2_{I,i}/\partial\sigma^2_i$ is the Jacobian of the transformation, $\hat{\sigma}^2$ is the implicit spot volatility found as $\sigma^2_I = h(\hat{\sigma}^2; \theta)$. Implicit volatility can be obtained as a limit of the following Newton–Raphson procedure:

$$\hat{\sigma}_i^2(p+1) = \hat{\sigma}_i^2(p) - \left[\frac{\partial h(\hat{\sigma}_i^2(p), \Theta)}{\partial \hat{\sigma}_t^2}\right]^{-1} [h(\hat{\sigma}_i^2(p), \Theta) - \sigma^2_{I,i}].$$

Even if in an absolute way, this procedure can be applied directly, it requires cumbersome charged CPU time since the transformation between Black–Scholes implied volatility and instantaneous volatility is non-linear and non-analytical. It is important to note that Pearsun and Sun (1994) and Duan (1994) succeeded in applying this econometric procedure since they used interest rate exponential *affine* model. In this special case, the transformation between an instantaneous interest rate and yield to maturity is very simple.

To obtain $\hat{\Theta}$ Renault and Touzi (1996) proposed an iterative procedure for implementation of the transformation between σ^2_I and σ^2, $\sigma^2_I = h(\hat{\sigma}^2; \theta)$, in the optimization of the log-likelihood function. The key point is that this estimation procedure provides simultaneously Hull and White's implicit volatilities and consistent estimators of the volatility process parameters. This is repeated until $\hat{\Theta}$ converges. More precisely, they introduced the following iterative procedure:

Step $2p$ $\quad \Theta^{(p)} \to \sigma_i^{(p+1)},\ i = 0, 1, \dots, n,$
Step $2p+1$ $\quad \sigma_i^{(p+1)},\ i = 0, 1, \dots, n \to \Theta^{(p+1)}.$

where step $2p$ is performed by solving a Hull and White's implicit volatility and step $2p + 1$ is the maximum likelihood estimate from data obtained by step $2p$.

In the special case where the at-the-money Black–Scholes implied volatilities are available, this iterative procedure is shown to be an EM (expectation-maximization) algorithm where the step $2p$ (step $2p + 1$) corresponds to step E (step M) of the EM algorithm. However, they noticed that the general properties of EM algorithms did not apply since the support of the latent variables given the observable ones depends on the unknown parameters. They argue that for a large enough sample size the algorithm converges *almost surely* towards the true value of the parameters. Furthermore as Renault–Touzi wrote, this procedure can be seen as correcting the approximating bias of the method used by Heynen, Kemna and Vorst (1994) who considered near-the-money, short maturity Black–Scholes implicit volatilities as proxies.

This iterative procedure reduces the charged CPU time dramatically because the numerical integration procedure has only to be called a fraction of the times as compared with directly maximizing the log-likelihood function where $\hat{\sigma}^2$ changes for every change in Θ. Nevertheless, the drawback of this procedure is that we do not get an explicit estimate of the volatility risk premium. Thanks to the Hull–White model which assumed a nonsystematic volatility risk this drawback is not important here. More importantly, Renault and Touzi (1996) pointed out that there were two asymptotic points of view relevant for this iterative procedure. The first one concerns the continuous time limit obtained by letting the time between observations go to zero (this is related to the near integrated time series). The second one consists of considering an infinite number of observations with fixed time between observations. Renault and Touzi (1996) recalled that except for the instantaneous variance parameter the maximum likelihood estimator does not converge with the true value of the parameters when the time between observations goes to zero, they considered the second asymptotic point of view.

Finally, the key point is that this estimation procedure provides simultaneously Hull and White's implicit volatilities "*and*" consistent estimators of the volatility process parameters. If at-the-money options are available at any time, this iterative procedure is shown to be an EM (expectation-maximization) algorithm, associated with the observations of Black–Scholes implied volatility, that converges *almost surely* towards the true value of the parameters. As Renault–Touzi pointed out a natural starting point of the iterative procedure is $\Theta^{(0)} = 0$. In fact, Table III and Figure 2 from the previous section reports the two first steps of Renault–Touzi (1996) iterative procedure ($p = 0$). Since $\Theta^{(0)} = 0$, it is clear that the first step filtered Hull–White's implicit volatilities equals to the market volatility index, VX1. Furthermore Renault–Touzi confirmed that this first iteration ($p = 0$) corresponding to the step 0 and 1 can be identified to the method used by Heynen, Kemna and Vorst (1994) who considered near-the-money, short maturity Black–Scholes implicit volatilities as proxies. Step two is obtained by solving a Hull and White's implicit volatility with the previous parameters. Step three should be able to estimate the parameters from data obtained by step $2p$ using a Maximum Likelihood procedure. However, from Figure 2 it should be noted that we cannot

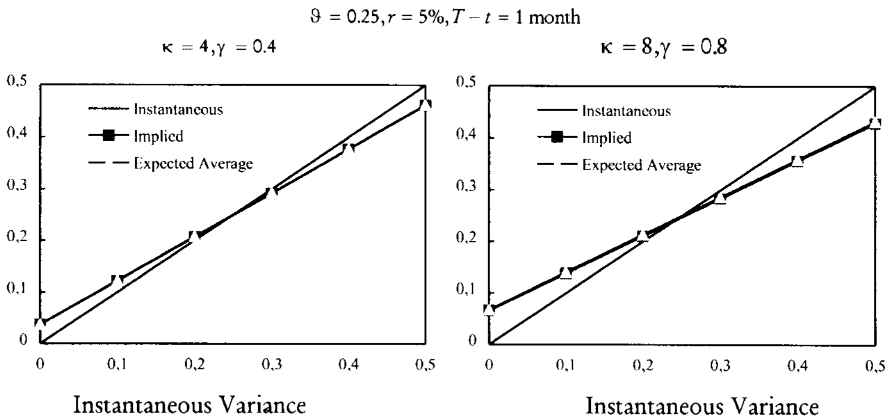

Figure 3. Expected average variance and instantaneous variance versus theoretical ATM implied variance ($\vartheta = 0.25$, r = 5%, $T - t = 1$ month).

obtain all implicit instantaneous volatilities based on the annualized French Market Volatility Index and the previous estimates of the parameters. Thus, in our case, Renault–Touzi's procedure does not permit us to obtain a whole implied volatility series and hence fails to provide parameters estimate.

Nevertheless, we suggest using Feinstein's (1992) methodology: he demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of the option. Following Ball and Roma (1994), it can be shown that:

$$\sigma_I^2 \approx E\left[\frac{1}{\tau}\int_0^{\tau}\sigma_u^2\,\mathrm{d}u\right] = \vartheta + (\sigma^2 - \vartheta)\frac{1 - e^{-\kappa\tau}}{\kappa\tau}.$$

From Figure 3, it can be infered, first, that Theoretical Implied Variance and Expected Average Variance seems to be very close and this justified Feinstein's (1992) approximation. Second, since Feinstein's definition assumed implicitly in this model that Theoretical Implied Variance is a affine function of the instantaneous one, it can be verified from Figure 3 that this seems to be true. Finally, it can also be shown that the relationship between Theoretical Implied Variance and Instantaneous Variance seems to be justified *only* when its level is near its long term value (the intersection point of Theoretical Implied Variance *and* the first bisectrice).

In this case,

$$\sigma_{I,i}^2 = \vartheta + (\sigma_i^2 - \vartheta)\frac{1 - e^{-\tilde{\kappa}\tau}}{\tilde{\kappa}\tau} \Leftrightarrow \sigma_i^2 = \vartheta + (\sigma_{I,i}^2 - \vartheta)\frac{\tilde{\kappa}\tau}{1 - e^{-\tilde{\kappa}\tau}},$$

σ_t^2 is therefore an *affine* function of the implied volatility. And the log-likelihood function is:

$$\mathcal{L}(\sigma_{I,n}^2, \ldots, \sigma_{I,0}^2; \Theta) = \sum_{i=1}^{n}\{-\ln|J_i| + \ln f(\hat{\sigma}_i^2|\hat{\sigma}_{i-1}^2; \Theta)\},$$

Table IV. Estimators based on market volatility index MONEP-VX1 ($d\sigma^2 = \kappa(\vartheta - \sigma^2)\,dt + \gamma\sigma\,dW_2(t)$ and $\sigma_I^2 \approx E[1/(T-t)\int_t^T \sigma_s^2\,ds]$

	Value	t-stat
κ	5.41814	3.68011
ϑ	0.063271	4.33034
γ	0.848799	30.0029
$\mathcal{L}$	3537.44	

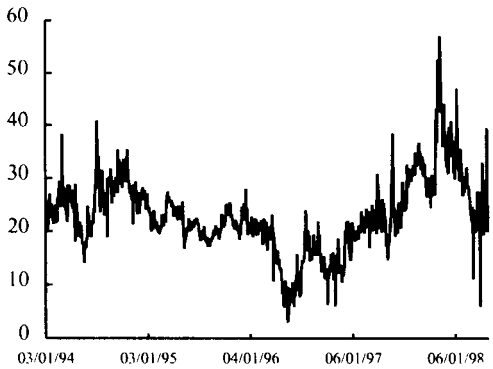

Figure 4. Implicit instantaneous volatility from market volatility index VX1.

where $J_i = 1 - e^{-\kappa\tau}/\kappa\tau$.

Note that if $\kappa(T-t)$ is low then $1 - e^{-\kappa(T-t)} \approx \kappa(T-t)$ and therefore $\sigma_I^2 \approx \sigma_t^2$ i.e., this methodology can be identified to the one used by Heynen, Kemna and Vorst (1994) who considered near-the-money, short maturity Black–Scholes implicit volatilities as proxies. In fact, like Renault–Touzi's iterative procedure, this can be seen as correcting the approximating bias of the method used by Heynen, Kemna and Vorst (1994).

The estimate for κ, the adjustment speed for σ^2, is 5.42, which implies a slower mean-reversion than that found in Table III. The half-life of the process is about one month and a half.

2.3. A MULTI HORIZON FORECAST OF STOCK MARKET VOLATILITY

Renault and Touzi (1996) pointed out that, in the Hull and White (1987) model, due to the increasing feature of the Black–Scholes formula with respect to the

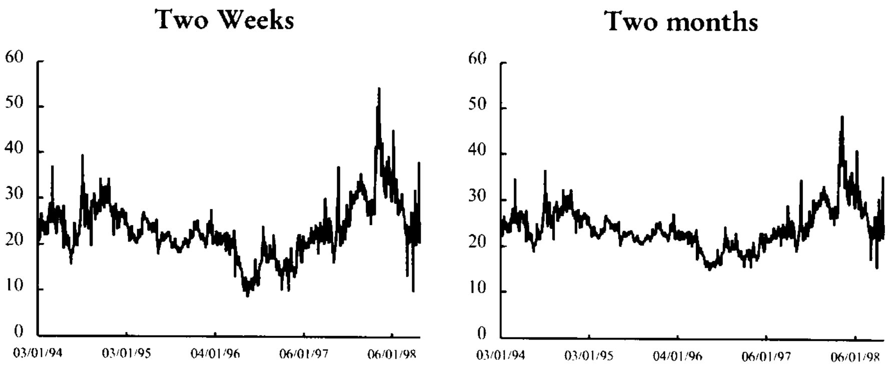

Figure 5. Volatility indexes issued from market volatility index VX1.

volatility parameter, a precise definition of the Black–Scholes' implied volatility can be given as the unique solution of:

$$\sigma_I^2(x, \sigma^2; \Theta) = h(x, \sigma^2; \Theta),$$

where $h = (C^{\mathcal{BS}})^{-1} \circ C$, $\Theta = (\kappa, \vartheta, \gamma)$, and σ is the instantaneous volatility. The implicit Hull and White's (1987) instantaneous volatility and the implied parameters from VX1 may allow us to calculate the implied volatility of an option no matter what maturity it is having. Here we create two new volatility indexes based on VX1: one with a maturity of two weeks whereas the other having a maturity of two months. These indexes are plotted in Figure 5.

Following the previous section, let $\sigma_t(\tau)$ be the realized volatility over the remaining contract life, measured from day t to day $t+\tau$ ($\tau = 2$ weeks or 2 months), defined as:

$$\sigma_t^2(\tau) = \frac{1}{2}\left(\frac{\Gamma\left(\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau}{2}\right)}\right)^2 \sum_{i=1}^{\tau}(R_{t+i} - \bar{R})^2,$$

where R_t is the return at t and $\bar{R}$ is the sample mean of R_t. Again, the predictive power of a volatility forecast can be estimated by regressing the realized volatility on forecast volatility:

$$\sigma_t(\tau) = a + b\hat{\sigma}_t + \epsilon_{t+1}, \tag{3}$$

where $\hat{\sigma}_t$ is the volatility forecast measured on day t, taken as the implied volatility and the historical rate is defined as $\sigma_{t-\tau}(\tau)$. Results are presented in Table V. The table shows that the two daily volatility indexes implied by VX1 contain information for future volatility over the remaining contract life.

Table V. Predictability regressions $\sigma_t(\tau) = \alpha + \beta\hat{\sigma}_t + \epsilon_{t+1}$

Intercept	Slopes on Implied	Historical	R^2
τ = 2 weeks			
0.004775*	0.612886*	0.225795	
(0.000878)	(0.092543)		
0.006672*		0.401140*	0.151908
(0.000610)		(0.053854)	
0.004370*	0.466862*	0.173605*	0.221965
(0.000882)	(0.108665)	(0.059802)	
τ = 2 months			
0.004058*	0.660926*		0.252267
(0.000934)	(0.090056)		
0.004862*		0.568199*	0.310523
(0.000815)		(0.077965)	
0.004025*	0.214538	0.437090*	0.319794
(0.000915)	(0.112903)	(0.103908)	

* Significantly different from zero at the 5 percent level.

3. Conclusion

The MONEP Market Volatility Index (VX1) is an average of CAC 40 option (PX1) implied volatilities. The findings reported in this study indicate that a strong relationship exists between VX1 and the future realized stock market volatility over a forecast horizon of one month. On the basis of these findings, and those reported in the study of Fleming, Ostdiek and Whaley (1995) for the CBOE Market Volatility Index (VIX), we apply the maximum likelihood estimation procedure on a VX1 time series to recover Hull and White's (1987) implicit instantaneous volatility. While the Renault and Touzi's (1996) statistical iterative procedure of filtering (of the latent volatility process) and estimation (of its parameters) failed to provide estimates of the parameters of the unobservable latent volatility process, we exploited Feinstein's (1992) research which demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of the option. In that case, since implied volatility is used in the same spirit as yield to maturity on the bond market, we applied direct maximum likelihood statistical inference on our analysis as Pearson and Sun (1994) and Duan (1994) had done in the case of interest rates. More importantly, we show how the index can be used to generate volatility forecasts over different horizons and that these forecasts are reasonably accurate predictors of future realized volatility.

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