Trading in Convergent Markets

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1 Introduction

The purpose of this paper is to find an optimal trading strategy for trading in stocks of two companies (or currencies of two countries) which will merge in the future (or join monetary union in the case of currencies). If the merger actually happens at time T, we assume that the ratio of the two stock prices S^1_T/S^2_T will be equal to some pre-specified constant C:

$$\frac{S^1_T}{S^2_T} = C. \tag{1}$$

The constant C is the ratio in which the two companies exchange their old stocks for the newly created merged company. Generally, there are many examples of convergent markets when there are two or more processes (stock prices, exchange rates or interest rates) for which we have some partial information about their relative future evolution in the above form.

Given the merger, the arbitrage free market would restrict the dynamics of the two stocks in such a way that no risk free profit is possible. However, in many cases the merger is often legally challenged and it is not obvious that it would indeed happen for sure. Therefore we can still observe uncertainty governing the prices of stocks of the two companies. In this sense, we are studying the situation of the conditional arbitrage: the risk-free profit is possible only if the merger actually takes place. Some authors refer to similar situations as statistical arbitrage.

In our paper, we assume that the two stocks are driven by two possibly correlated Brownian motions. We show that the condition (1) is equivalent to the restriction on the final position of the linear combination of the two Brownian motions. It is very similar to the Brownian bridge process, with the exception that it is two dimensional. We refer to it as a planar Brownian bridge process. First, we study the correlation structure and stochastic differential equation governing such a process. Next, we apply these results to the dynamics of the two convergent stocks. An investor trading in these stocks can obtain optimal trading strategy which maximizes his expected profit. Interestingly enough, this optimal strategy is often different from simply locking the arbitrage opportunity.

1.1 Brownian Bridge

Suppose that the underlying process is a Brownian Bridge with the starting point $X_0 = a$ and the end point $X_T = b$. The dynamics of such process can be written in the following form:

$$dX_t = \frac{b - X_t}{T - t} dt + dW_t, \quad X_0 = a. \tag{2}$$

2 Planar Brownian Bridge

2.1 Distribution of Planar Brownian Bridge

In this subsection, we will use the general result valid for jointly normal vectors (X_1, X_2). Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right). \tag{3}$$

Then the conditional distribution of X_1 given X_2 is

$$(X_1 | X_2) \sim N(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}). \tag{4}$$

We can use this result to get the conditional distributions of W_t^1 and W_t^2 given that $a_1 W_T^1 + a_2 W_T^2 = b$. We easily conclude that for $s < t$,

$$(W_s^1, W_t^1 | a_1 W_T^1 + a_2 W_T^2 = b) \sim N\left(\begin{pmatrix} \frac{a_1 b \cdot s}{(a_1^2+a_2^2)T} \\ \frac{a_1 b \cdot t}{(a_1^2+a_2^2)T} \end{pmatrix}, \begin{pmatrix} s - \frac{a_1^2 s^2}{(a_1^2+a_2^2)T} & s - \frac{a_1^2 st}{(a_1^2+a_2^2)T} \\ s - \frac{a_1^2 st}{(a_1^2+a_2^2)T} & t - \frac{a_1^2 t^2}{(a_1^2+a_2^2)T} \end{pmatrix} \right), \tag{5}$$

$$(W_s^2, W_t^2 | a_1 W_T^1 + a_2 W_T^2 = b) \sim N\left(\begin{pmatrix} \frac{a_2 b \cdot s}{(a_1^2+a_2^2)T} \\ \frac{a_2 b \cdot t}{(a_1^2+a_2^2)T} \end{pmatrix}, \begin{pmatrix} s - \frac{a_2^2 s^2}{(a_1^2+a_2^2)T} & s - \frac{a_2^2 st}{(a_1^2+a_2^2)T} \\ s - \frac{a_2^2 st}{(a_1^2+a_2^2)T} & t - \frac{a_2^2 t^2}{(a_1^2+a_2^2)T} \end{pmatrix} \right), \tag{6}$$

and

$$(W_s^1, W_t^2 | a_1 W_T^1 + a_2 W_T^2 = b) \sim N\left(\begin{pmatrix} \frac{a_1 b \cdot s}{(a_1^2+a_2^2)T} \\ \frac{a_2 b \cdot t}{(a_1^2+a_2^2)T} \end{pmatrix}, \begin{pmatrix} s - \frac{a_1^2 s^2}{(a_1^2+a_2^2)T} & -\frac{a_1 a_2 st}{(a_1^2+a_2^2)T} \\ -\frac{a_1 a_2 st}{(a_1^2+a_2^2)T} & t - \frac{a_2^2 t^2}{(a_1^2+a_2^2)T} \end{pmatrix} \right). \tag{7}$$

In terms of the correlation structure, we have

$$\rho^{11}(s,t) = \mathbb{E}(W_s^1 W_t^1) = s \wedge t - \frac{a_1^2 st}{(a_1^2 + a_2^2)T}, \tag{8}$$

$$\rho^{22}(s,t) = \mathbb{E}(W_s^2 W_t^2) = s \wedge t - \frac{a_2^2 st}{(a_1^2 + a_2^2)T}, \tag{9}$$

$$\rho^{12}(s,t) = \mathbb{E}(W_s^1 W_t^2) = \frac{-a_1 a_2 st}{(a_1^2 + a_2^2)T}. \tag{10}$$

Now the question is what kind of dynamics would match the above correlation structure.

2.2 Dynamics of PBB

Theorem 2.1 *Suppose that W_t^1, W_t^2 are two Brownian motions with $W_0^1 = W_0^2 = 0$, $dW_t^1 \cdot dW_t^2 = \rho dt$ and suppose that*

$$a_1 W_T^1 + a_2 W_T^2 = b. \tag{11}$$

Than the dynamics of W_t^1, W_t^2 can be written as

$$dW_t^1 = (a_1+\rho a_2)\tfrac{b-a_1W_t^1-a_2W_t^2}{(T-t)(a_1^2+2\rho a_1a_2+a_2^2)}\ dt + \tfrac{a_1+\rho a_2}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^1 - \tfrac{a_2\sqrt{1-\rho^2}}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^2, \quad (12)$$
$$dW_t^2 = (a_2+\rho a_1)\tfrac{b-a_1W_t^1-a_2W_t^2}{(T-t)(a_1^2+2\rho a_1a_2+a_2^2)}\ dt + \tfrac{a_2+\rho a_1}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^1 + \tfrac{a_1\sqrt{1-\rho^2}}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^2, \quad (13)$$

where B_t^1 and B_t^2 are two independent standard Brownian motions.

Proof. Let us introduce two new processes:

$$U_t^1 = a_1W_t^1 + a_2W_t^2, \quad (14)$$
$$U_t^2 = -(a_2+\rho a_1)W_t^1 + (a_1+\rho a_2)W_t^2. \quad (15)$$

It is easy to show that U_t^1 and U_t^2 are independent, U_t^1 is a Brownian bridge starting at 0 and ending at b, and U_t^2 is a Brownian motion. Thus we can write dynamics of U_t^1 and U_t^2, which turns out to be

$$dU_t^1 = \frac{b-U_t^1}{T-t}dt + \sqrt{a_1^2+2\rho a_1a_2+a_2^2}dB_t^1, \quad (16)$$
$$dU_t^2 = \sqrt{1-\rho^2}\sqrt{a_1^2+2\rho a_1a_2+a_2^2}dB_t^2, \quad (17)$$

where B_t^1, B_t^2 are independent standard Brownian motions. If we express these equations in terms of W_t^1 and W_t^2, we get

$$d(a_1W_t^1+a_2W_t^2) = \tfrac{b-a_1W_t^1-a_2W_t^2}{T-t}\ dt + \sqrt{a_1^2+2\rho a_1a_2+a_2^2}dB_t^1, \quad (18)$$
$$d(-(a_2+\rho a_1)W_t^1+(a_1+\rho a_2)W_t^2) = \sqrt{1-\rho^2}\sqrt{a_1^2+2\rho a_1a_2+a_2^2}dB_t^2. \quad (19)$$

Now we can solve for dW_t^1 and dW_t^2 to obtain

$$dW_t^1 = (a_1+\rho a_2)\tfrac{b-a_1W_t^1-a_2W_t^2}{(T-t)(a_1^2+2\rho a_1a_2+a_2^2)}\ dt + \tfrac{a_1+\rho a_2}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^1 - \tfrac{a_2\sqrt{1-\rho^2}}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^2, \quad (20)$$
$$dW_t^2 = (a_2+\rho a_1)\tfrac{b-a_1W_t^1-a_2W_t^2}{(T-t)(a_1^2+2\rho a_1a_2+a_2^2)}\ dt + \tfrac{a_2+\rho a_1}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^1 + \tfrac{a_1\sqrt{1-\rho^2}}{\sqrt{a_1^2+2\rho a_1a_2+a_2^2}}\ dB_t^2. \quad (21)$$

3 Stock Merger

Theorem 3.1 *Assume that the dynamics of two stock prices is given by*

$$dS_t^1 = S_t^1(\mu_1dt + \sigma_1dW_t^1), \quad (22)$$
$$dS_t^2 = S_t^2(\mu_2dt + \sigma_2dW_t^1), \quad (23)$$

where W_t^1 and W_t^2 are two correlated Brownian motions with $dW_t^1 \cdot dW_t^2 = \rho dt$. Moreover, if the condition

$$S_T^1 = C \cdot S_T^2 \quad (24)$$

is given, then the dynamics of stock prices can be expressed as

$$\frac{dS_t^1}{S_t^1} = \frac{\sigma_1(\sigma_1-\rho\sigma_2)\log\left(CS_t^2/S_t^1\right)+(T-t)\left(\mu_1\sigma_2(\sigma_2-\rho\sigma_1)+(\mu_2+\frac{1}{2}\sigma_1^2-\frac{1}{2}\sigma_2^2)\sigma_1(\sigma_1-\rho\sigma_2)\right)}{(T-t)(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2)}dt$$
$$+\frac{\sigma_1(\sigma_1-\rho\sigma_2)}{\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}}dB_t^1+\frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}}dB_t^2\ , \quad (25)$$

$$\frac{dS_t^2}{S_t^2} = \frac{\sigma_2(\sigma_2-\rho\sigma_1)\log\left(S_t^1/CS_t^2\right)+(T-t)\left(\mu_2\sigma_1(\sigma_1-\rho\sigma_2)+(\mu_1+\frac{1}{2}\sigma_2^2-\frac{1}{2}\sigma_1^2)\sigma_2(\sigma_2-\rho\sigma_1)\right)}{(T-t)(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2)}dt$$
$$-\frac{\sigma_2(\sigma_2-\rho\sigma_1)}{\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}}dB_t^1+\frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sqrt{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}}dB_t^2\ , \quad (26)$$

where B_t^1 and B_t^2 are two independent standard Brownian motions.

PROOF. For geometric Brownian motion we have

$$S_t^i = S_0^i \exp((\mu_i - \tfrac{1}{2}\sigma_i^2)t + \sigma_i W_t^i). \quad (27)$$

Therefore the condition $S_T^1 = C \cdot S_T^2$ translates as

$$\sigma_1 W_T^1 - \sigma_2 W_T^2 = \left(\mu_2 - \mu_1 - \tfrac{1}{2}(\sigma_2^2 - \sigma_1^2)\right) T + \log\left(\tfrac{CS_0^2}{S_0^1}\right). \quad (28)$$

This identity defines planar Brownian bridge with

$$a_1 = \sigma_1,$$

$$a_2 = -\sigma_2,$$

and

$$b = \left(\mu_2 - \mu_1 - \tfrac{1}{2}(\sigma_2^2 - \sigma_1^2)\right) T + \log\left(\tfrac{CS_0^2}{S_0^1}\right).$$

The dynamics of the planar Brownian bridge was given in Theorem 2.1 in terms of two independent standard Brownian motions B_t^1 and B_t^2. Plugging expressions (12) and (13) in (22) and (23), we conclude the proof.

3.1 Special Cases

One important special case is when we have just a single convergent stock, i.e., condition $S_T^1 = C$ is given. This is just a special case of the above analysis with $S_0^2 = 1$, $\sigma_2 = \mu_2 = 0$. Substituting back to (25), we conclude:

Corollary 3.2 *Suppose that*

$$dS_t = S_t(\mu_1 dt + \sigma_1 dW_t^1), \quad (29)$$

and S_T is known. Then

$$dS_t = S_t\left(\left(\frac{\log S_T - \log S_t}{T-t} + \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dB_t^1\right). \quad (30)$$

Remark 3.3 *Notice that the actual drift μ_1 has absolutely no relevance to the dynamics of the stock price if we know its terminal price.*

4 Optimal Strategy

Suppose that there is a trader who trades in the two stocks of companies which will merge and their stock prices evolve according to (25) and (26). The wealth of the investor Y_t^q follows

$$dY_t^q = q_t^1 dS_t^1 + q_t^2 dS_t^2, \tag{31}$$

where q_t^1 and q_t^2 are investor's positions in the market. The objective of the investor is to maximize his expected wealth (or utility of his expected wealth) subject to some trading constraints. We might assume that his absolute number of shared purchased is limited (say $|q_t^i| \leq 1$, or $|q_t^1 S_t^1| + |q_t^2 S_t^2| \leq K$), or we may assume that his wealth gets never below to some specific level ($Y_t \geq L$ for all t).

Let's assume for the following analysis that the investor wants to maximize the expected profit resulting from his trading subject to the trading constraint $|q_t^i| \leq 1$. We have the following result in this case:

Theorem 4.1 *The optimal strategy maximizing $\mathbb{E}Y_T^q$ subject to $|q_t^i| \leq 1$ is given by*

$$q_t^1 = \operatorname{sgn}(\sigma_1 - \rho\sigma_2) \cdot \operatorname{sgn}\left(\frac{CS_t^2}{S_t^1} - R_t^1\right), \tag{32}$$

$$q_t^2 = \operatorname{sgn}(\sigma_2 - \rho\sigma_1) \cdot \operatorname{sgn}\left(\frac{S_t^1}{CS_t^2} - R_t^2\right), \tag{33}$$

where

$$R_t^1 = \exp\left((T-t)\left[-\tfrac{1}{2}(\sigma_1^2 - \sigma_2^2)) - \frac{\mu_1\sigma_2(\sigma_2-\rho\sigma_1)}{\sigma_1(\sigma_1-\rho\sigma_2)} - \mu_2\right]\right), \tag{34}$$

$$R_t^2 = \exp\left((T-t)\left[-\tfrac{1}{2}(\sigma_2^2 - \sigma_1^2) - \frac{\mu_2\sigma_1(\sigma_1-\rho\sigma_2)}{\sigma_2(\sigma_2-\rho\sigma_1)} - \mu_1\right]\right). \tag{35}$$

PROOF. We can write

$$\mathbb{E}Y_T^q = Y_0 + \mathbb{E}\int_0^T (q_t^1 dS_t^1 + q_t^2 dS_t^2)dt \tag{36}$$

$$= Y_0 + \mathbb{E}\int_0^T q_t^1 \frac{\sigma_1(\sigma_1-\rho\sigma_2)\log\left(CS_t^2/S_t^1\right)+(T-t)\left(\mu_1\sigma_2(\sigma_2-\rho\sigma_1)+(\mu_2+\frac{1}{2}\sigma_1^2-\frac{1}{2}\sigma_2^2)\sigma_1(\sigma_1-\rho\sigma_2)\right)}{(T-t)(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2)}dt$$

$$+\mathbb{E}\int_0^T q_t^2 \frac{\sigma_2(\sigma_2-\rho\sigma_1)\log\left(S_t^1/CS_t^2\right)+(T-t)\left(\mu_2\sigma_1(\sigma_1-\rho\sigma_2)+(\mu_1+\frac{1}{2}\sigma_2^2-\frac{1}{2}\sigma_1^2)\sigma_2(\sigma_2-\rho\sigma_1)\right)}{(T-t)(\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2)}dt. \tag{37}$$

The investor then chooses long position if the dt term is positive, or short position if it is negative. The signum of the dt term is given by formulas (32) and (33), which concludes the proof.

4.1 Special Cases

Corollary 4.2 *When we know that $S_T = C$, the optimal strategy reduces to*

$$q_t^* = \operatorname{sgn}(\log(\tfrac{S_T}{S_t}) + \tfrac{1}{2}\sigma^2 \cdot (T-t)). \tag{38}$$

PROOF. Immediately follows from the expression (30).

Remark 4.3 *Notice that this strategy* q_t^* *is different from the strategy which locks the arbitrage coming from the knowledge of the terminal stock price* S_T*. For example if* $r = 0$*, then in order to lock the arbitrage, we would sell the stock when* $S_t > S_T$*, and we would buy the stock if* $S_T > S_t$*, which would guarantee us the profit* $|S_T - S_t|$*. But the strategy which maximizes the expected profit* q^* *suggests that we should buy the stock when* $S_T \cdot \exp(\frac{1}{2}\sigma^2 \cdot (T-t)) > S_t$ *and sell the stock otherwise. This is rather surprising, because when* $S_T \cdot \exp(\frac{1}{2}\sigma^2 \cdot (T-t)) > S_t > S_T$*, one should buy the stock in order to maximize the expected profit, even knowing that the stock price would drop.*